ΣΟΦΙΑ–SOPHIA

Is the square a rectangle?

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Abstract

The concept of a geometric object is mediated, among other things, by the experiences that we have with it. The hypothesis of this study is that to perform a process of conceptualization of an object, using a dynamic geometry environment, it helps not only to formulate, formalize and structure hierarchical and economic definitions of objects, but also to make ostensible the definition with which students work in a context of demonstrative activity. In order to support the hypothesis, we analyzed class sessions of two consecutive academic spaces of an initial teacher training program, and a questionnaire was applied to the students. Evidence was thus sought that, based on the theoretical framework that guided the study, could determine the impact of working in an environment where dynamic geometry software is used as mediator for learning. The results allowed to establish, among other things, that work done with dynamic geometry must be accompanied by intentional actions guided by the teacher. It was evidenced that, even after using dynamic geometry, there were kept present several difficulties with respect to the concept of the square geometric object, such as the predominance of the figurative aspects over the conceptual ones and the difficulty to modify personal definitions of the concept, and the conceptual images that arise when students perform a demonstrative activity.

Key words: Conceptualization, construction of definitions, dynamic geometry, hierarchical definitions, learning.

Introduction

The construction of definitions in school mathematics has been a relatively well studied problem in mathematics education. Regarding theoretical aspects, researchers such as Tall and Vinner (2002) formulated their conceptualization of *concept image* and *concept definition*, Fischbein (1993) proposed for geometry the theory of figurative concepts, and de Villiers (1998) a categorization of how definitions can be introduced in the classroom, whether constructive or descriptive. He also establishes the type of definitions that students can construct, whether they are hierarchical or not, economic or not. In the field of applied studies, production is even more extensive, and reviewing it exceeds space in an article of this nature. It is relevant for this work to mention that since the emergence of dynamic geometry software, studies have been developed to determine its relationship with the construction of hierarchical classifications (de Villiers 2004, Jones 2000), or to test whether students elaborate conceptual definitions or simple representations (Furinghetti 2002). The aforementioned authors provide important elements for studies interested in performing an interpretive, non-cognitive analysis of what happens in the geometry classroom when working with definitions.

In the present article, we report a study developed to verify the hypothesis: "the use of dynamic geometry has a significant impact on the conceptualization process." The main objective of the study was to analyze if the use of dynamic geometry facilitates the construction and formulation of hierarchical and economic definitions. The context of the research was an activity carried out around two particular concepts, rectangle and square, in two geometry courses during two consecutive semesters, to have the same population of an initial training program for mathematics teachers, of The Mathematics Degree from the Universidad Pedagógica Nacional. In these courses, it had been elaborated a design for the conceptualization supported in dynamic geometry.

In the first course, the emphasis is primarily put on the study of definitions; and skills are developed so that, using dynamic geometry, somebody can perform conjecture processes where from the exploration of a situation, a conjecture is established; in the second course, the purpose is the formulation of demonstrations of conjectures established with the use of dynamic geometry. The introduction of the use of dynamic geometry was, at that moment, an innovation. Therefore, these courses became the propitious scenario to validate our hypothesis. To verify it, we made audio and video records of the classes of the two mentioned spaces. In addition, we had students' responses, in the second semester, to a questionnaire (see section 3.3.2) designed to examine aspects that were not visible in the audio and video recordings of interactions. Based on the previous theoretical review, categories of analysis were built, especially supported by de Villiers (1986, 1998, 2004). With these as a framework of analysis, the interactions and the answers given by the students to the questionnaire were studied.

Theoretical perspective

Researchers such as Tall and Vinner (2002) have studied the process of constructing definitions; their hypothesis is that for students to access mathematical definitions, it is necessary to emulate the historical process: to move from the construction of definitions in empirical contexts to the establishment of a formal definition through refinement. For students, the conflict between the approximations and the formal definition constitutes a real obstacle that may influence the understanding of the concept. Thus, the way in which the conceptualization process is performed is a crucial aspect in the mathematical education of a person.

For Tall and Vinner (2002), the process of forming a concept involves the interaction between the *definition of the concept*, which corresponds to that given from mathematics and the *conceptual image*; or the definition existing in the student's mind, which does not always coincide with that of the concept. Such interaction takes place over a long period during which the experiences with the concept must be transforming the conceptual image and the personal definition. The results of their research show that, usually, teachers believe that students form the *conceptual image* through the *definition of the concept*. Consequently, they expect that the latter will control the former, and that in any task with the use of the concept, students will resort to the definition of the concept. Practice has shown that the path followed is different, because students generally only use their *conceptual image*.

Fischbein (1993) introduces the term *figural concept* to emphasize the dual nature of concepts in geometry, since they involve both theoretical and figurative aspects (usually associated with conceptual images). Usually it happens that the figural prevails over the conceptual; that is, that the *conceptual image* prevails over the *definition of the concept*, which may explain many errors in the geometric reasoning of a student (Mariotti & Fischbein, 1997; Fischbein, 1993).

De Villiers (1986, 1998) characterizes the process of constructing definitions; for him, there are two processes associated with the task of defining concepts in mathematics: *descriptive* (a posteriori) and *constructive* (a priori). *Descriptive definitions* are achieved when experiences have been made for some time with the properties of the object; and from these, there are chosen those from which the others can be checked or deduced logically. The subset thus determined constitutes the definition, and the other properties become theorems. The role of these definitions is to systematize existing knowledge. *Constructive definitions* arise when a given property in a definition is changed by some logical process (exclusion, generalization, specialization, replacement or addition of properties to the definition) to form a new concept; its role is to produce new knowledge.

De Villiers (1998) presents a categorization for the elaboration of classifications associated to the process of defining: *hierarchical* and *partitioning*. *Classifying hierarchically* means organizing concepts in such a way that the more particular are subclasses of more general ones (class inclusion). In a *partitioning classification* the different subclasses of the concept are considered as disjunctive of each other. For example, in the first case, a square can be defined as a particular case of rectangle, and this in turn as a particular parallelogram; in the second, a square is not a rectangle and a rectangle is not a parallelogram. Hierarchical classifications are usually linked to definitions that contain only the sufficient and necessary properties to define the object, called by de Villiers as *correct and economical definitions*. Partial classifications are often associated with definitions which, while not incorrect, contain nonessential information, which de Villiers names as *correct non-economic definitions*.

Another construct that played an important role in our study is that of *demonstrative activity*. According to Perry, Camargo, Samper and Rojas (2006), this goes beyond demonstration and involves two processes: "One conformed by actions tending to produce a conjecture and another conformed by actions tending to produce a justification" (p. 397). In the first one, actions such as visualization, exploration, guesswork, and verification have an important role; the second involves actions such as explaining, justifying and systematizing results.

The definitions of geometric objects play an important role in the demonstrative activity because, as reported by several researchers, one of the difficulties associated with the deductive process lies in the poor understanding of the nature and role of definitions, and in the difficulty to differentiate the sufficient and necessary conditions of the same. If a personal *definition of the concept* does not match the *definition of the concept*, the possibility of success in justification will be affected.

With regard to the use of dynamic geometry in the process of defining, we coincide with Mariotti (1997) and Govender (2002) in considering that a dynamic geometry environment allows that the figurative constraints, that is, those that are mistakenly assigned to the definition by the limitation of its figurative representation, arise at the time of constructing or validating it. Thus, using dynamic geometry in the framework of a process of conceptualization can help overcome the difficulties that the restricted and prototypical representations usually generate.

For Mariotti, the images produced in the micro world of a program of dynamic geometry are controlled in a logical way by the commands of the different menus; that is to say, in the figure there are both precept-like and logical components linked to the figurative and conceptual aspects of the object. Therefore, dynamic geometry is very useful, not only for the dialectical dilemma to be given between them, but also to achieve a proper integration of the same within the processes related to logical reasoning. From Govender, we return to his didactic proposal in which the definitions are not provided by the teacher, but obtained after a constructive process and a creative activity that contributes to the understanding of the use and the role of the definitions. The exploration of figures in a dynamic geometry environment allows that, when using the drag, the invariants of the object be revealed in order to determine the necessary and sufficient conditions that really allow defining it. In addition, Furinghetti & Paola (2002) consider that the construction work in dynamic geometry allows to make ostensive the definition that the students use.

Materials and methods

The present study approaches "from the didactic and pedagogical context" the intervention of a teacher in a teaching and learning space where, from the theoretical references presented above, it is analyzed the potential of using a mediator such as dynamic geometry, and the appropriate intervention of a teacher in the conceptualization of the square geometric object.

We characterize this study as descriptive and interpretive, with an emergent design model (Calvo 2001), because there are analyzed not only the written productions of students, but also their verbal interactions in the classroom. It is an emerging design because the instruments for data collection were designed in the course of the work, and the categories of analysis to examine the results, framed in the theoretical aspects that served as reference, were designed by the authors. The quantitative elements are simple counts of frequencies that serve as reference for qualitative analysis.

The data of the study were obtained during the development of the Elements of Geometry and Plane Geometry, directed by the same teacher, courses that are part of the geometry line of the initial teacher training program of the Universidad Pedagógica Nacional of Colombia. Follow-up was done to the same population (25 students aged 16 to 20 years) for one year. The two courses were chosen since in the first one, it is worked on the process of constructing definitions using dynamic geometry; and in the second one, these (definitions) are used in the framework of the demonstrative activity.

In the course *Elements of geometry*, there were taken audio and video records of 7 classes in which students developed three activities: 1. Defining the rectangle, 2. The rectangle and the square and 3. The rectangle and parallelogram. In the Plane Geometry course, there were taken audio and video records of 5 classes, in which the students developed a set of activities around a situation that is named "The Saccheri quadrilateral." In the same course, it was applied a questionnaire with five questions related to the square geometric object. From the transcription of the audio and video of the classes, a qualitative analysis was made in light of the theoretical frame of reference. The answers to the questionnaire questions were classified according to the categories that are formulated below, (which were) established in the light of the theoretical framework.

Table 1 lists the general aspects of the analysis, the instruments for collecting information, the purpose of collecting the data, and the investigative action on the data produced. see next page.

In order to avoid losing continuity between data and their analysis, in the results and discussion sections are posed the questions related with reports on the findings from the audio and video records of three activities of the course of *Geometry Elements* and the situation of the course of *Plane Geometry*, together with their objectives and what they showed.

The questions proposed and their intentionality were as follows:

Question one: *For each question, determine whether the answer to the question is Yes, No, or Do not know*:

a) ABCD is a parallelogram. Is it a rectangle?

b) ABCD is a rhombus. Is it a rectangle?

We tried to establish if the students have constructed *hierarchical* or *partitioning* definitions of the rectangle with respect to the ones of parallelogram and square, and if they are *economic*.

Question two: Write down as many definitions as you can for 'square', and explain why you know that each one defines 'square'.

It is sought, on the one hand, to establish whether the students formulate, among those who propose them, an *economic* and *hierarchical* definition in which they establish the necessary and sufficient conditions; on the other hand, we want to analyze the variety of definitions presented, since they are linked to the *personal space of examples*. Requesting several definitions, and not just one, opens the possibility that some are not *economic* and *hierarchical*, and that there emerge some *personal*

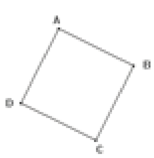
Table 1. Moments of analysis and instruments of data collection

Moments	Instruments	Objective	Investigative action on the information
1. Process of introduction of the definition	Audio and video records of seven classes of Elements of Geometry, (first semester) in which the students developed three activities around the definition of square.	Characterize the conceptualization process of the squared geometric object.	Qualitative analysis of the transcripts, in the light of the theory that frames the investigation
2. Use of the concept in the context of the demonstration	Audio and video records of five classes of Plane Geometry (second semester) in which the students approached the development of the situation "The Saccheri quadrilateral"	Analyze conceptual images and personal definitions of square. Analyze the concept used in the context of demonstrative activity.	
3. Application of the questionnaire	Questionnaire to students of Plane Geometry (5 questions)	Evaluate the process retrospectively, and determine if there is influence of the use of dynamic geometry.	Classification of responses in the light of the established categories as a result of the theoretical framework. Simple count of category frequency for a global analysis of results

Source: self-made

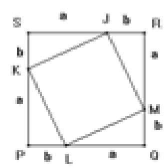
definitions of the concept that show the impact of the figural representations of the same. Finally, we wanted to find evidence of the impact of the use of dynamic geometry in the construction of the definition of square.

Question Three: On the screen of a calculator you can see the following figure: What kind of quadrilateral is ABCD and how can you insure it?



This question is proposed in the context of dynamic geometry, with the hope that students directly mention its use, in order to obtain evidence of the possible impact on the formation of the *conceptual image*. It is sought to determine if the figural representation suggests to the student a specific relation between the *definition of the concept*, the *personal definition of the concept* and the *personal image of the concept*. In addition, we want to detect the knowledge of students about the conditions required to establish if the representation, in a dynamic geometry environment, is that of a square; they should refer to the definition and mention that the perpendicularity and the congruence of segments must be kept under the drag.

Question four: Data: PQRS is a square. The points J, K, L, M determine segments on the sides, as in the figure, of lengths **a** and **b**. What are the key steps to demonstrate that the JKLM quadrilateral is a square?



In particular, this problem is an adaptation of one presented by Moise & Downs (1986). The intention is to analyze the concordances and discrepancies between the *conceptual image*, the *personal definition of the concept* and the *concept used* with the *definition of the concept* when the students use it in the process of justification.

Categories of analysis

Based on what Villiers (1986, 1998, 2004) and Govender (2002) proposed, there were established categories of analysis that allowed the interpretation of the justifications and to constitute approaches about the relationship between *conceptual image* and *concept definition*. The criterion of *hierarchy* is considered present if it is expressed any relation of correct inclusion of the geometric objects rectangle and square among them, and with respect to the parallelogram. For the *economy* criterion, it is considered that it is present if there is any kind of relationship between the invariants of the concept that makes that the conditions expressed in the definition be minimal; for example, when it is expressed in response to question 4 of the questionnaire: "*it would look like a square*" and it is argued that "*I would assure myself by taking the measure of the sides AB CD and the angle ABC*", the answer is in the same category, regarding hierarchy, but not economy, of the one expressed by Adriana to the same question: "If I determine with a calculator the parallelism between its opposite sides, I take the distance between each one of the vertices with the option distance and length to verify the congruence between sides; likewise, I ensure that AB is perpendicular to AD with the option perpendicularity, and if it is so, I can affirm that they determine a right angle and therefore the others are right angles too. With a good use of this process, I can determine if it satisfies the conditions that at first sight led me to determine the figure either as a rectangle, square or rhombus."

"Quadrilateral with four congruent sides and four right angles" and "quadrilateral with four right angles" will be considered as *non-economic* nor *hierarchical* definitions of square and rectangle, respectively; since during the course, a rectangle was defined as a quadrilateral with three right angles; and a square, as a quadrilateral with three right angles and two adjacent congruent sides. A definition is *hierarchical* if it presents the inclusion of the square in the rectangles, and *economic* if it reduces the conditions in the number of congruent sides or of right angles. The established categories are:

Hierarchical, economic and expressing the definition (HED): In the expressed justification there are elements to affirm that the conceptual image of rectangle or square is associated with a *hierarchical* and *economic* definition; and explicit definitions of these quadrilaterals are presented. An example is the justification of Nora when defining the relation between parallelogram and rectangle in question 1 (b): "*For a quadrilateral to be a rectangle, it must be a parallelogram and one of its internal angles is right, and in this case we have the first condition but not the second one, so it may have a right angle or it may not, so it may be or may not be a rectangle*."

Economic and hierarchical (EH): The expressed justification gives elements to affirm that the conceptual image of rectangle is associated with a *hierarchical and economic definition*, but it does not explicitly present a definition of rectangle or square; an example is the justification of Lucas to question 1 (b): "It would be needed a right angle to (be able to) determine it".

Non-economic hierarchical (NEH): In the justification, there are elements to affirm that the conceptual image of rectangle or square is associated to a *hierarchical* definition, and it presents information that indicates that the *conceptual image* of these concepts obeys to a *non-economic definition*. An example is Orlando's justification in question 1 (b): "Because every rectangle is a parallelogram, but not every parallelogram is a rectangle. Since a parallelogram has two pairs of parallel sides as well as the rectangle, but a parallelogram doesn't need to have right angles, while a rectangle does."

Simply hierarchical (SH): Elements are offered to affirm that the *conceptual image* of rectangle or square is associated to a *hierarchical* definition, but it does not provide information that allows to affirm that its *conceptual image* is associated to an *economic definition* or not. The explanation of Francisco in 1 (b) is an example: "*Since the definition of a parallelogram is a quadrilateral with both pairs of opposite sides (being) parallel, and this leaves room to consider many quadrilaterals with these properties, the rectangle is only one of them*."

Simply economic (SE): The expressed definition is *economic* if it establishes sufficient and necessary conditions to define the object, but it does not allow to establish whether it is *hierarchical* or not. Dora gives an example for square when expressing: "*Quadrilateral with all its congruent sides and a right angle*."

It does not contribute information (NA): In the expressed justification, there are no elements to affirm that its conceptual image of rectangle or square is associated with a *hierarchical* or *economic* definition, or no justification is given. The definition of Johann for rectangle, in answer 1 (b), illustrates the above: "*We could only say that there are four points joined by right lines. To be a rectangle, it must have four right inner angles*."

No hierarchical or economic but correct (NHEC): In the justification, there are no elements to claim that their *conceptual image* of rectangle or square is associated with a *hierarchical* or *economic* definition with respect to the object mentioned in the question, but they cite a correct definition. An example is the one expressed by Patricia in defining square: "*Quadrilateral with 4 right angles and all its congruent sides*."

Results

This section illustrates the interactions among the students and with the teacher in each of the activities mentioned in the previous section. It shows a route in the development of the conceptualization that the students express about the square and the rectangle, emphasizing what the teacher does, so that this development is supported in the dynamic geometry. The process, as formulated in our hypothesis, should lead to the formulation of hierarchical and economic definitions.

Process of introduction of the definition

As it was mentioned in the previous section, in this section we present and analyze the results of the audio and video recordings of the three activities of the *Geometry Elements* course.

Defining the rectangle

The following task was designed to be addressed with dynamic geometry:

i) *Perform a graphic representation on rectangle paper.*

ii) *Construct the figure in the calculator*[1].

iii) *Write down the definition.*

It was sought that students explained their *personal definitions of the rectangle concept*, and the *conceptual image* that they have of it, through a figural representation of the geometric object. We wanted to determine how they use the elements expressed in its definition to construct a rectangle with dynamic geometry; that is, if it makes its definition clear.

Table 2 reports the definitions of rectangle given by some groups of students, selected by the teacher to promote the discussion in class.

Table 2. Definitions of 'rectangle' given by student groups

Group	Definition
1	Flat figure with two pairs of parallel sides, and two non-adjacent congruent sides.
2	Flat figure, closed with four segments. Each pair of opposing segments are congruent. It has four right angles.
3	Quadrilateral, with two pairs of congruent segments, and they have right angles.
4	Quadrilateral with two pairs of parallel and congruent sides, and their adjacent sides are perpendicular.

Source: self-made

Everybody refers to the congruence of the pairs of opposite sides, the presence of right angles, and the non-congruence of the adjacent sides, which corresponds to the prototypical representation of the object that excludes the square as a type of rectangle (Fischbein, 1993; de Villiers, 2004). Groups 1 and 2 excluded the word quadrilateral without realizing that its definition can refer to figures with more sides; that is, they establish necessary but not sufficient conditions, which may reflect a misunderstanding of the role of definitions in mathematics (Zaskis and Leikin, 2008).

During the development of the *Elements of Geometry* course, it was used dynamic geometry, but this is the first task in which it is examined the correspondence between the written definition and the performed construction. Table 3 shows how the teacher emphasized the correspondence that must exist, making it clear the definition that underlies the construction (that was) elaborated by one of the groups. Thus, it is sought to support the understanding of what a mathematical definition is. see next page.

The subject was discussed regarding the hierarchical inclusion of the square as a rectangle when a student questions one of the proposed definitions. Next we present the protocols of the class where the definition was discussed.

Julian: But quadrilateral with four right angles and a pair of opposite sides congruent; it may refer to a square, right? It would be necessary to specify many other things to arrive at the correct definition of rectangle, because that definition itself could be confused with the one for square.

Teacher: If I have a figure that meets those conditions, could it be square?

Julian: Yes, that's what I'm saying.

Teacher: What is being said about the relationship between square and rectangle?

Julian: That the two of them have four right angles. That a square is a rectangle.

Table 3. Correspondence Action

Definition	Description of the construction	Teacher's interpretation of the construction-definition relation
Flat figure, closed with four segments. Each pair of opposing segments are congruent. It has four right angles.	Mercedes: First, we did what they put as [segment] BD [referring to the one made by another group], we did a perpendicular straight line, and we started to put perpendicular [to segment] BC, perpendicular [to segment] CA.	So the definition does not match what you did. You did not make a pair of opposing congruent segments [pointing to the text of the definition]. You did not make four ... ah, you made only three right angles.

Source: self-made

Teacher: It depends on how you define rectangle. If I define it this way, it would seem, according to the image we have, that a square is a rectangle. If you don't want the square to be a rectangle, then change the definition.

As it was evidenced, Julian clearly considers that the set of rectangles and that of squares are not disjunctive, but it is explicitly stated his image of a *partitioning* relation (de Villiers, 2004) between these two figures: "*He would have to specify many other things to arrive at the correct definition of rectangle*."

The rectangle and the square

This task also required the use of dynamic geometry: *For each numeral, construct a quadrilateral and with drag, make the figure to fulfill the required condition. Decide if the figure is a square.*

1. Quadrilateral with four congruent segments.

2. Quadrilateral with four segments that determine four right angles.

3. Quadrilateral with four congruent segments that determine four right angles.

4. Quadrilateral with four congruent segments and one right angle.

The activity sought to determine which properties are sufficient and which are necessary to define a square; to discuss the definition, partitioning or hierarchical, of a square (de Villers, 1998) with respect to a rectangle; and to give rise to the economic definition of each geometric object.

The above activity addresses the economy in definitions. After establishing that four congruent sides and four right angles are conditions that define a square, we studied the situation of four congruent segments that was accepted by some, product of the drag in a soft construction (Healy 2000). In the next episode, the teacher presents a counterexample to discard it, using dynamic geometry, and an example that emphasizes the necessary and sufficient conditions for the figure to be a square.

Teacher: The idea was to start with any quadrilateral, to drag until the sides were congruent (she projects the construction). At that point, they are more or less congruent; sometimes the measurements are not exact. How many (students) say it's a square? Many said yes, but over there I have an example of a figure that has all its four sides congruent and is not a square. So what happens? Notice that I can drag (in the figure) so that they become, not only the congruent sides, but that become perpendicular (she drags the construction), which was what they did; they put two conditions on it, (and then) they dragged it, making it to comply them (the conditions).

Finally, it arises the discussion about the hierarchy of definitions of rectangle and square, an issue that was not initially clear to students, as it can be seen in the next episode of the class. It is important to remember that students are not yet working with a geometric axiomatic system, and they only have an informal approach to geometric concepts and facts.

Dora: Professor, you were saying that the square is also a rectangle.

Teacher: It depends on the definition given. If I define rectangle as: quadrilateral with four right angles; and square as: quadrilateral with four right angles and four congruent sides; we would see that the square is a rectangle to which I have put other conditions. But if I define square as: quadrilateral with four congruent sides and a right angle, I cannot say that the square is a rectangle, even though we were already convinced that these conditions forced the four angles to be right; but I cannot prove it.

Jaime: To say something, in the rectangle we write (that) it is a quadrilateral with four right angles and its adjacent sides are unequal.

Teacher: Here comes the question, if I add here unequal adjacent sides (she writes on the board), then the square is not a rectangle. When we give the definition, if we want the squares to be a subset (of the rectangles), then the unequal sides condition must be removed.

This type of task is based on the support that constitutes the dynamic geometry for the acceptance of a hierarchical classification for quadrilaterals (de Villers, 1998).

The rectangle and the parallelogram

The third task was the analysis of the constructions proposed by the students in the following situation for the dynamic geometry environment: *Construct a quadrilateral that satisfies the following conditions. Determine if each one of them can be considered as a definition of a rectangle.*

a) *Parallelogram with a right angle.*

b) *Quadrilateral with a pair of consecutive right angles and a pair of opposing congruent sides.*

The objective was to see how the use of dynamic geometry enhances the process of conceptualization and the identification of relevant properties. This activity comes to be a *constructive definition*, (de Villiers, 2004), since it addresses the reformulation of the definition of rectangle as a class of parallelogram; that is, it seeks to establish a hierarchical definition of a rectangle with respect to the parallelogram.

There are discussed two aspects of working with definitions in a context of dynamic geometry use. First, the teacher analyzes the construction-definition correspondence; second, she expresses the need for empirical evidence and the need for a reference definition. The teacher, as evidenced in the next section of the class, emphasizes the need to make a robust construction (Healy, 2000) that meets the conditions (keep under the drag), and not a representation that perceptually fulfills the conditions (soft construction) as they did.

Teacher: When working with definitions, just as we are doing, they're not really tied to anything theoretical. The definition can be any one that gives us those sufficient conditions that we know a figure must have, in order to be a rectangle. We all have an idea of what a rectangle is, because it is not the first time we work with it. The first one said: parallelogram with a right angle. So when I look into your calculators to see what you have done, what should I see? ... What I do in the calculator is to uncover all that you have hidden, in order to see how the construction was. If you say parallelogram, I start with a segment AB and I know that I must construct a parallel to it, because, by definition, I need two pairs of parallel opposite sides. But I know that I need a right angle, and since I cannot use the drag, because the drag meets the conditions momentarily, I have to make a robust construction that maintains the conditions that I am asking for. I must draw the perpendicular (she traces the perpendicular *a* trough *A*) and you draw the segment that you want. But you must have two pairs of parallel opposite sides. So what's next?

After the intervention of the teacher, and once the construction is achieved with the established conditions, she asks the students for the conditions to determine if the constructed quadrilateral is actually a rectangle, and she shows how to determine whether what is reported as acceptance or not of the definition was based on information obtained from the construction. Next are reported the dialogues with the students:

Teacher: I have: a right angle and a pair of parallel opposite sides. But I said parallelogram, this requires two pairs of opposite parallel sides, that is to say, that I lack the parallel *a* through *B*. And then I complete the figure. Because the only way we have to see if it's a rectangle is ... how?

Several (students): Measuring.

Teacher: Measuring the angles, why? (Asking for an explanation to a student.)

Carla: Because we have a definition; rectangle: quadrilateral with four right angles.

Teacher: It's the only thing we have to use; that definition. This is ready (she points the vertex A). Then, what did they have to do? ... Measure and measure again (she writes 90 at each vertex) and then use the drag. But I did not see those measures…

Julián: Teacher, and over there, with the drag, does it keep the 90 degrees?

Carla mentions the established definition for rectangle; in her *personal definition of the concept*, there is a change regarding the definitions previously expressed by the students; she stops being descriptive to focus on the conditions that must be met. When Julian inquires about whether the characteristics remain under the drag, he shows some confidence in the empirical evidence provided by software tools. This part of the task confirms that the work in dynamic geometry contributes to the construction of hierarchical definitions in quadrilaterals (de Villiers, 2004). In this case, it is because the students construct the conditions that determine a parallelogram, and add conditions to obtain a rectangle.

Use of concept in demonstration context

A problem situation was introduced in the Plane Geometry course, in which the Saccheri quadrilateral is described, when congruence of triangles and inequalities in triangles have already been approached. The Parallel Postulate has not yet been enunciated and the theorems related to parallelism have not been demonstrated. The intention of this situation is to confront the student with the impossibility of proving that the quadrilateral is a rectangle, if such postulate has not previously been established. The problem was:

“Given the quadrilateral ABCD, being right angles C and D, and segment AD is congruent with segment BC. What can be said about angles A and B?

Students are asked to construct, with dynamic geometry, the described quadrilateral. With this, the teacher seeks to stimulate the study of quadrilaterals, the definitions and theorems related to them. As it was foreseeable, the first statement of the students, at seeing the representation of the quadrilateral of Saccheri, was to say that it was a rectangle. But proving that implies using the definition of rectangle. The objective of our analysis was to confront the conceptual image of the student with the definition of the concept. Thus the teacher begins, as it is presented in the following extract of the class, discussing the definition of rectangle, to be able to advance on the hierarchical and economic definitions.

Teacher: We were analyzing a figure that had the following characteristics: it was a quadrilateral, being C and D right angles, and segment AD congruent with segment BC. And from that figure, we come to several conjectures. One was that angle B was congruent with angle A. Right? Among other things, what kind of figure does it produce?

Several (students): A rectangle

Teacher: Why?

Sandra: A quadrilateral with four right angles

Mercedes: Three, three.

Teacher: Three, why?

Mercedes: Because if there are three, the other one is already right.

Teacher: And how do we know it? As far as I remember, we had defined four, a quadrilateral with four right angles. Why do you say three, Mercedes?

Mercedes: I think the calculator showed us that three angles were enough; that four angles were not necessary.

Teacher: Last semester (evoking what was addressed in the course of Geometry Elements) we had defined rectangle with four right angles, but … Mercedes is right. We concluded that every time we have three right angles, it gives us a rectangle, and we decided to transform the definition and say that rectangle (she writes on the board) is a quadrilateral with three right angles. But the figure I gave you has two right angles. So why do you tell me it’s a rectangle?

Nora: We do not know which figure it is; we are checking it out.

Teacher: So, what we achieved is to show that we had right and congruent angles C and D; and are you going to prove that the other angles are right?

The previous intervention constitutes explicit evidence of the result of the process of construction of a definition with dynamic geometry, since the student was emphatic in affirming that three right angles were enough to define a rectangle, obtaining a more economic definition. This allows us to conclude that the use of dynamic geometry in conceptualization processes affects the *personal definition* of students. Defining a rectangle in this way reduces the conditions to be validated in the course of the demonstration.

Questionnaire

Question 1 (a):

From the results obtained in Table 4, it cannot be clearly inferred if the students use the definition in the justification because it was not explicitly requested to give a definition. We present in the table, as an example, two of the justifications given by the students with their respective categorization. In the first one, the personal definition of the concept is explained, but not in the second one. It is appreciated that students pose a sufficient and necessary condition for a parallelogram to be a rectangle, and they can establish relationships between the definitions of the involved objects by distinguishing sufficient and necessary conditions. This relates to the *personal space of examples*, insofar as together with the figurative representations, it allows the students to provide examples and non-examples of a geometric object. It is concluded that the activities developed in relation with the construction of definitions have had a positive impact on structuring their formal thinking, since one of the essential aspects in understanding the definition of a concept is the possibility of offering examples and non-examples thereof. Table 4 reports the definitions found and their categorization, as well as some specific examples of them that illustrate what was found in the answers to question 1 (a).

Table 4. Categories of definitions given by students and examples of them

Cat.	S	%	Student	Definition	Category
JED	5	20	Roberto	A parallelogram is a quadrilateral with two pairs of parallel sides, but not necessarily two adjacent sides form a right angle; and a rectangle is a parallelogram with at least one right angle.	JED
EJ	11	44			
SJ	5	20	Julián	Because to be a rectangle, it would need a right angle; and the definition of parallelogram only requires to be quadrilateral with two pairs of nonconsecutive parallel sides	EJ
JNE	2	8			
NA	2	8			

Source: self-made

Question 1 (b)

For this question, it was assumed a hypothesis of sustained interpretation in de Villiers (2004) related to the acceptance of a hierarchical classification when working in a dynamic geometry environment: if there are arguments in the presented justification that allow to recognize elements of economy or hierarchy (EJ, JED or SJ), then there is indirect evidence of the contribution of dynamic geometry in expanding the *space of examples* and the *conceptual image* of rhombus. For economy, we have that if it is rhombus and it has a right angle, then it is a square; although this can be deductively verified by making use of theorems and parallelism postulates, quadrilaterals and congruence, it is considered that by the way in which the activities were developed, the argumentation is more likely to come from the evocation of the activity with dynamic geometry than from the deductive process within the developed axiomatic system. In Table 5, we present two illustrative examples of the definitions given by students, as well as a general classification of them from the proposed categories.

Table 5

Student	Answer	Justification
Andrea	No idea	It depends on the angle because if it has a right angle, then it would be a square, and the square is a rectangle.
Lola	No idea	It's affirmative if the rhombus has at least a right angle, because ABCD would be a square and every square is a rectangle; but it would not be a rectangle if none of its angles is right. But, since it says nothing of the angles, and by the definition of a rhombus, we can only deduce that ABCD is a parallelogram, so we do not know whether ABCD is a rectangle or not.

Category	JED	EJ	SJ	NJEC	NA
Student	1	5	8	5	6
%	4	20	32	20	24

Source: self-made

The fact that only six students gave in their justification an argument that shows the presence of an *economic definition* (whether *hierarchical or not*) seems to be associated with a difficulty related to the personal definition of rhombus; if this does not correspond to the *definition of the concept*, the possibility of establishing a hierarchy between rhombus and square, and equally, determining the necessary and sufficient conditions for a rhombus to be a square, is truncated. This is seen in some of the answers and justifications given in the table, as it is the case of Roberto:

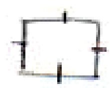
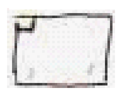
Student	Answer	Justification
Roberto	No	A rhombus is a parallelogram with two pairs of adjacent congruent sides, and a rectangle does not have those characteristics.

Source: self-made

Question 2

The students gave a total of 86 definitions; all of them delivered more than one definition, of which at least one was correct. The categorization is shown in Table 6.

Table 6. Categorization and examples of definitions given for a square

Student	Definitions	Explanation
Amanda	D1: Quadrilateral with 4 congruent sides	S1: It is the first notion of square, as a figure with all its sides (being) congruent.
	 D2: Quadrilateral with a right angle, so it has four right angles (congruent)	S2: Because when constructing a closed polygonal figure, starting from an angle
		

Cat.	JED	SE	NJEC	NA	JNE	Total
No.	38	10	19	17	2	86
%	44	12	22	20	2	

Source: self-made

50 definitions had characteristics of economy and/or hierarchy, and 19 were *correct definitions*, even though they did not reflect conditions of economy or hierarchy; this is not significant since the statement proposed to state: "*All possible definitions*." 17 were incorrect, a fact that can be explained with two reasons: first, the force of the figural representation of square that has been formed in the first years of schooling, as Amanda expresses it by saying: "because it is the first notion of square;" second, the rooting of an incomplete or incorrect definition of the geometric object that persists in the student's conceptions and that is evoked when they have to provide "*the possible definitions of;*" a finding consistent with Zaskis & Leikin (2008).

Some of the definitions, as reported in Table 7, reflect the particular way how there were developed the activities of constructing definitions in the dynamic geometry environment, which is evident in the explanations given by some of them:

Table 7. Evidence of the influence of dynamic geometry on the formulation of the definition.

	Definitions	Explanation	Cat.
Andrea	T1: Quadrilateral with four congruent sides and a right angle.	S1: Because we checked with the calculator and the created axiomatic system that it is needed only one right angle so that the other three ones are also right angles.	SE
	T2: Rectangle with 2 consecutive congruent sides	S2: Being rectangle with the following condition, the other sides are congruent	JED
	T3: Quadrilateral with 3 congruent sides and 1 right angle	S3 (no explanation)	SE

Source: self-made

Question 3

It is evident that the evoked object is influenced by the figural representation. Since the formation of a *personal definition* of the concept, that be closer and consistent with both the *conceptual image* and the *definition of the concept*, is a process mediated by the time of experience with the object, it can be explained that students evoke not only a square or a rhombus, but also rectangles, parallelograms, etc. The most evoked concept was the square, and solid arguments are given to justify that it is, reflecting the relation between a figural representation and the *evoked concept*, which makes that the *conceptual image* and the *personal definition* determine the process of argumentation and justification.

The difference between the students who determine a possible geometric object and those who determine it with certainty lies on the type of definition that they give. Students who present the square as the possible object showed elements of economy in their given definitions, as inferred from their answers and justifications (Lola's case). Most of the students who say that the represented object is a square give definitions in which the economy criterion does not appear (Nora's case). Examples of these definitions are given in Table 8.

Table 8. Definitions of the geometric object (appearance vs. reality)

Student	Quadrilateral	Actions to ensure it in Dynamic Geometry	Cat.
Lola	It seems to be a square	To check that it is, (based on D2 of question 2: Parallelogram with a right angle and two adjacent congruent sides.), I would go to the 'check parallelism property' menu and look at <B <<C and <C <<D. If so, also checking 'property', I would look if BC < CD, in which case <C would be straight and I would already know that ABCD is a square. Also, I could measure the 4 angles and see if they measure 90. If so, then I would measure the four sides of the quadrilateral to see if they are congruent, but it would take me longer. Or I could measure the angles and check the property of parallelism.	JED
Nora	It is a square	I take the measure of the sides to check for congruence, and check that the opposite sides are parallel; finally, I measure the internal angles to check that they are straight, or check that the adjacent sides are perpendicular.	JNE

Source: self-made

In this question, the hypothesis of interpretation is that those students who show economy in the actions to be performed consider unnecessary the exhaustive revision for the fulfillment of the definition of square. This may be due to the work with dynamic geometry and to the way how it was developed the construction activity for the definition of square. Table 9 shows the results obtained in this question.

Table 9. Objects evoked for question 3, quantity and categorization of them

Object(s) Evoked	S	%
Square	11	44
Parallelogram	1	4
Quadrilateral	2	8
Rhombus	1	4
Rectangle, square or rhombus	3	12
Parallelogram, square or trapezium	1	4
Parallelogram, square or rhombus	1	4
Parallelogram, square or rectangle	1	4
Parallelogram, rectangle, square or rhombus	3	12
No answer	1	4

Objects (amount)	S	%
One object (possible)	6	24
One object (certainty)	6	24
Three objects	5	20
Four objects	5	20
No answer	3	12

Cat.	JED	EJ	JNE	SJ	NJEC	NA	SE
Student	1	9	2	2	4	3	4
%	4	36	8	8	16	12	16

Source: self-made

In the description of the actions to be performed in a dynamic geometry (DG) environment, it became clear the difference between those who consider the dynamic aspect of the software and those who (only) see a static drawing on the screen. The explicit allusion to the drag shows that they think of a dynamic environment in which the apparent properties of figural representation can be verified, and the importance of those that are invariant to drag; this is evidenced in the answers of Diego and Julián, reported in table 10, which unfortunately do not allow us to infer their definitions of the geometric objects.

Table 10. Actions proposed by students in a DG environment

Diego	A parallelogram (perhaps with some right angles).	The graph does not insure anything, and it might not meet any conditions when dragging with the calculator.
Julían	A quadrilateral without any special characteristic.	I would measure the sides of the angles and would look if under the drag, the conditions are kept, or now what conditions it meets; in order to see what kind of quadrilateral it is, according to them.

Source: self-made

Some students made explicit the drag action as an essential aspect of their argumentation, as Patricia points out, which is reported in table 11 in the actions to be carried out:

Table 11. Explanation of the drag action to check properties

Patricia	She doesn't postulate the type of quadrilateral, but in her development task, she reported having evoked the square.	I would check in the calculator that AB, BC, CD and DA are equal to each other, and I would measure <BAD or <ADC or <DCB or<CBA; I would say it is a square if the measures of the sides are equal, and if any of the angles is right, and when moving the points, they do not change because of the drag.

Source: self-made

In spite of the above, it cannot be stated conclusively that the presence of economy or hierarchy in the personal definitions of the concept is associated with the use of dynamic geometry or with the use of drag. However, because of the results obtained in the analysis of question 1 (a), it can be inferred that leading the students to think about an environment where they can perform certain actions and verify properties allows them to elaborate definitions that reflect economy and hierarchy aspects.

Question 4

Since key steps were to be taken to prove that it was a square, the definition should be evoked in a context of use. It was desirable that this be economical, since it would reduce the length of the demonstration. On the other hand, a hierarchical and economic definition, which would seem better at first, would lead to a more complicated process. Evoking an economic definition can be the result of the process of constructing the definition with dynamic geometry, because of the way in which the definition was presented in the courses, particularly in the one of Plane Geometry. This can be seen by comparing the steps proposed by Orlando and Patricia, who evoke an economic and hierarchical definition

It was evidenced an economic definition of square in only 10 responses, which shows that the main difficulty associated to the definition is given in the use.

Student	Synthesis of key steps presented by students	Cat.
Orlando	He uses the definition of square and concludes that KJ < JM <ML <LK are congruent because there are respective rectangular triangles (with) congruent sides (criterion LAL applies). He proposes to use the measure of the angles to show that any angle of the inner quadrilateral is right, and concludes "for definition of square, we have a square [referring to JKLM.] (It shows four congruent sides and a right angle).	SE
Patricia	For definition of square, the four angles of PQRS are straight. For the theorem of Pythagoras, we arrive at the congruence of the sides of the quadrilateral JKLM. For LLL, all the angles are congruent; and for the definition of congruent triangles and the theorem of the sum of the internal angles of a triangle, it can be shown that <KJM is right. Since the four angles of JKLM are right, for theorem of Internal angles between parallels and supplementary angles, JKLM is a parallelogram; and because it is a parallelogram and it has a right angle and two consecutive congruent sides, JKLM is a square.	JED

Source: self-made

Discussion of results

In relation with the partial results, most of them were presented in the previous section, given the nature of the research carried out and the actions inherent to the course's methodology. In a global way it can be said that there were evidenced some significant advances towards the conceptualization of the square, but also towards the formulation of hierarchical and economic definitions. However, it should be emphasized that for teaching purposes, the two aspects may not always be desirable or relevant, especially when the definitions are used in a demonstration context.

Conclusions

About building definitions

The conceptualization process used in the courses is supported by two relevant references in the field of research in didactics of geometry; first, that dynamic geometry makes it possible to highlight the *personal definition of the concept*; second, that the activity of defining can be developed in a *constructive or descriptive* way (de Villiers, 2004), according to specific didactic intentions.

The use of dynamic geometry contributes to the elaboration, acceptance and use of *hierarchical and economic definitions* for square and rectangle. The design of the activities provided empirical evidence about the reduction of conditions for defining an object; it can be seen how adding a property modifies the object, showing that some are subsets of others (de Villiers 1986, 1998).

The analysis allows us to see that the hierarchical and economic definitions of square can be influenced by the conceptual image of other geometric objects such as the quadrilateral, rectangle or rhombus, which are usually associated with canonical and prototypical representations. Dynamic geometry alone will not influence the process of conceptualizing geometric objects. Only the action of the teacher, supported by the use of the software, can provide conceptual elements that help the *personal definitions of the concept* and the *conceptual images* of the same approach the *definition of the concept*.

Among the actions to be highlighted and that are evidenced in the registers, we can mention, on the one hand, the way in which the activities were designed, with a plan focused on clear objectives, raised from the conception of what should be the initial training of teachers in the geometry line; on the other hand, there is the attentive attitude of teachers to relevant instances of the class in which it is promoted the evocation of activities carried out in the dynamic geometry environment, not only to question *personal definitions*, but also to relate the *definition of the concept* in the processes of argumentation with these definitions and with the *conceptual images*.

It was evidenced the construction of *hierarchical definitions* between different geometric objects, rectangle with respect to parallelogram, rectangle with square, parallelogram with square and rhombus with square. This last hierarchy was the least evidenced, possibly because it is associated with a difficulty with the *personal definition* of rhombus, or the persistence of figural aspects on conceptual ones. The diversity of definitions for the same geometric object and the acceptance of their equivalence by students, from a theoretical perspective, is an important result of the process of constructing definitions, since it is usual to implicitly accept the uniqueness of the definitions of mathematical objects, and not to question about it.

On the persistence of the figurative aspects

The process developed with the support of the software of dynamic geometry helps to harmonize the conceptual and conceptual aspects in a coherent whole, and that the figural eventually becomes subordinated to the conceptual. For this to be achieved, it is important that the teacher emphasizes the dynamic nature of the software, in particular the dragging actions of the elements of the object or the construction of the same; this is because the properties invariant to the drag make up the necessary and sufficient conditions for the definition of the object. The mastery of the figurative aspects is difficult to overcome, which was evidenced in the prototypical representations used by the students.

Difficulty to modify personal definitions of the concept and conceptual images was evident; this is probably because previous student experiences have led to the establishment of erroneous or incomplete definitions of geometric objects. This may be explained because, just as the acquisition of a concept is a process mediated by experiences over time with the object, and it is even more for the modification or reification; they also require the possibility of expanding and enriching the field of experiences with the object. One way to do this is by using dynamic geometry.

On the use of definitions in the contexts of demonstrative activity

The analysis of question 4 of the questionnaire showed that the main difficulty in the deduction process lies in the use of definitions. This is because the concept used may be distant from the *definition of the concept*, either because the *personal definition of the concept*, if it does not coincide with the *definition of the concept*, ends up imposing itself; or because the figural predominates over the conceptual. The aspects of economy and hierarchy, although desirable from a mathematical point of view, may not be it from the didactic point of view, even for argumentation processes. This emerged in the same question 4 where assuming a *hierarchical definition* of square versus parallelogram makes the demonstration more costly, even though doing so would indicate a better understanding of the nature of geometric objects and of the definition itself. Ideally, the student can adapt a definition to the context.

Footer

1 The calculator has the Cabri Geometry software, but it can still be performed on a laptop or other device, and with a different dynamic geometry software, such as *Geogebra*.

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